

# LOCAL GRADIENT ESTIMATES FOR DEGENERATE ELLIPTIC EQUATIONS

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**ABSTRACT.** This paper is focused on the local interior  $W^{1,\infty}$ -regularity for weak solutions of degenerate elliptic equations of the form  $\operatorname{div}[\mathbf{a}(x, u, \nabla u)] + b(x, u, \nabla u) = 0$ , which include those of  $p$ -Laplacian type. We derive an explicit estimate of the local  $L^\infty$ -norm for the solution's gradient in terms of its local  $L^p$ -norm. Specifically, we prove

$$\|\nabla u\|_{L^\infty(B_{\frac{R}{2}}(x_0))}^p \leq \frac{C}{|B_R(x_0)|} \int_{B_R(x_0)} |\nabla u(x)|^p dx.$$

This estimate paves the way for our forthcoming work [6] in establishing  $W^{1,q}$ -estimates (for  $q > p$ ) for weak solutions to a much larger class of quasilinear elliptic equations.

## 1. INTRODUCTION

Consider the Euclidean space  $\mathbb{R}^n$  with integer  $n \geq 1$ . Denote  $B_R(x) = \{y \in \mathbb{R}^n : |y - x| < R\}$  and  $B_R = B_R(0)$ . In this paper we investigate local gradient estimates for weak solutions to equations of divergence form

$$(1.1) \quad \operatorname{div}[\mathbf{a}(x, u, \nabla u)] + b(x, u, \nabla u) = 0 \quad \text{in } B_3,$$

where the vector field  $\mathbf{a}$  and the function  $b$  satisfy certain ellipticity and growth conditions. Specifically, let  $\mathbb{K} \subset \mathbb{R}$  be an interval, and let  $\mathbf{a} = (\mathbf{a}^1, \dots, \mathbf{a}^n) : B_3 \times \mathbb{K} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $b : B_3 \times \mathbb{K} \times \mathbb{R}^n \rightarrow \mathbb{R}$  be Carathéodory maps such that  $\mathbf{a}$  is differentiable on  $B_3 \times \mathbb{K} \times (\mathbb{R}^n \setminus \{0\})$ . We assume also that

$$(H1) \quad \mathbf{a}(x, z, 0) = 0 \quad \forall (x, z) \in B_3 \times \mathbb{K};$$

and there exist  $p > 1$  and  $\gamma_0, \gamma_1 > 0$  such that

$$(H2) \quad \sum_{i,k=1}^n \frac{\partial \mathbf{a}^i(x, z, \eta)}{\partial \eta_k} \xi_i \xi_k \geq \gamma_0 |\eta|^{p-2} |\xi|^2 \quad \forall (x, z, \eta, \xi) \in B_3 \times \mathbb{K} \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^n;$$

$$(H3) \quad \sum_{k=1}^n \left| \frac{\partial \mathbf{a}(x, z, \eta)}{\partial \eta_k} \right| \leq \gamma_1 |\eta|^{p-2} \quad \forall (x, z, \eta) \in B_3 \times \mathbb{K} \times (\mathbb{R}^n \setminus \{0\});$$

$$(H4) \quad \sum_{i=1}^n \left| \frac{\partial \mathbf{a}}{\partial x_i}(x, z, \eta) \right| + |\eta| \left| \frac{\partial \mathbf{a}}{\partial z}(x, z, \eta) \right| \leq \gamma_1 (|\eta|^{p-1} + |\eta|^p) \quad \forall (x, z, \eta) \in B_3 \times \mathbb{K} \times \mathbb{R}^n;$$

$$(H5) \quad |b(x, z, \eta)| \leq \gamma_1 (|\eta|^{p-1} + |\eta|^p) \quad \forall (x, z, \eta) \in B_3 \times \mathbb{K} \times \mathbb{R}^n.$$

We would like to stress that (H1)–(H5) are only assumed to hold for  $z \in \mathbb{K}$  which might be a strict subset of  $\mathbb{R}$ , and the constants  $\gamma_0, \gamma_1$  can depend on  $\mathbb{K}$ . For example, in some

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cross-diffusion equations in population dynamics (see [5] and the references therein), we have  $p = 2$ ,  $\mathbf{a}(x, z, \eta) = (1 + z)\eta$ , and  $\mathbb{K}$  is a bounded subset of  $(0, \infty)$ .

A weak solution  $u(x)$  of (1.1) is defined to be a function in  $W_{\text{loc}}^{1,p}(B_3)$  that satisfies  $u(x) \in \mathbb{K}$  for a.e.  $x \in B_3$ , and

$$-\int_{B_3} \mathbf{a}(x, u, \nabla u) \cdot \nabla \varphi(x) dx + \int_{B_3} b(x, u, \nabla u) \varphi(x) dx = 0 \quad \forall \varphi \in W_0^{1,p}(B_3) \cap L^\infty(B_3).$$

The equations of the form (1.1) have been studied extensively in the literature, see [3, 4, 7–14]. In particular, interior  $C^{1,\alpha}$  regularity for homogeneous  $p$ -Laplace equations was established by Uraltceva [14], Uhlenbeck [13], Evans [4] and Lewis [8]. Regarding the local regularity for general quasilinear equations (1.1), the following classical result is proved by DiBenedetto [3] and Tolksdorf [12].

**Theorem 1.1.** ([3, Theorem 1], [12, Theorem 1]) *Assume (H1)–(H3), and*

$$(H4') \quad \sum_{i=1}^n \left| \frac{\partial \mathbf{a}}{\partial x_i}(x, z, \eta) \right| + \left| \frac{\partial \mathbf{a}}{\partial z}(x, z, \eta) \right| \leq \gamma_1 |\eta|^{p-1},$$

$$(H5') \quad |b(x, z, \eta)| \leq \gamma_1 |\eta|^p$$

*hold for every  $(x, z, \eta) \in B_3 \times \mathbb{K} \times \mathbb{R}^n$ . If  $u$  is a bounded weak solution of (1.1), then  $u \in C_{\text{loc}}^{1,\alpha}(B_3)$  and there exists a constant  $M > 0$  depending only on  $n, p, \gamma_0, \gamma_1$  and  $\|u\|_{L^\infty(B_3)}$  such that*

$$(1.2) \quad \|\nabla u\|_{L^\infty(B_2)} \leq M.$$

Our purpose is to explicate estimate (1.2), namely, to bound the local  $L^\infty$ -norm of  $|\nabla u|$  by its local  $L^p$ -norm that preserves the scaling in  $x$ . Our achieved result holds for more general vector field  $\mathbf{a}(x, u, \nabla u)$  and function  $b(x, u, \nabla u)$  than the ones required in Theorem 1.1. Precisely, we obtain:

**Theorem 1.2.** *Assume that (H1)–(H5) hold. Let  $u$  be a weak solution of (1.1) that satisfies*

$$(1.3) \quad \|u\|_{L^\infty(B_{\frac{1}{4}})} \leq M_0.$$

*Then there exists  $C > 0$  depending only on  $n, p, \gamma_0, \gamma_1$  and  $M_0$  such that*

$$(1.4) \quad \|\nabla u\|_{L^\infty(B_{\frac{R}{2}}(x_0))}^p \leq \frac{C}{|B_R(x_0)|} \int_{B_R(x_0)} |\nabla u(x)|^p dx, \quad \forall x_0 \in B_1, \quad 0 < R \leq 1.$$

When the growths of  $\mathbf{a}$  and  $b$  in the  $\eta$  variable are weaker, the assumption (1.3) on the local boundedness of the solution can be dropped. In particular, we obtain the following result when conditions (H4) and (H5) are strengthened appropriately.

**Theorem 1.3.** *Assume (H2)–(H3), and*

$$(1.5) \quad \sum_{i=1}^n \left| \frac{\partial \mathbf{a}}{\partial x_i}(x, z, \eta) \right| + |\eta| \left| \frac{\partial \mathbf{a}}{\partial z}(x, z, \eta) \right| + |b(x, z, \eta)| \leq \gamma_1 |\eta|^{p-1}, \quad \forall (x, z, \eta) \in B_3 \times \mathbb{K} \times \mathbb{R}^n.$$

*Then there exists  $C = C(n, p, \gamma_0, \gamma_1) > 0$  such that for any weak solution  $u$  of (1.1), the estimate (1.4) holds true.*

Gradient estimates of the type (1.4) were discovered by Uhlenbeck [13] for elliptic systems of the form  $\operatorname{div}(\mathbf{A}(|\nabla u|^2) \cdot \nabla u) = 0$ , and were later extended further by Tolksdorf [11] for a larger class of quasilinear elliptic systems. In [3, Proposition 3.3], DiBenedetto derived estimate (1.4) for weak solutions to scalar equation  $\operatorname{div} \mathbf{a}(\nabla u) = 0$ . The same estimate was established in [1, Lemma 1.1] for equations of the form  $\operatorname{div}(|\nabla u|^{p-2} \nabla u) + b(x, u, \nabla u) = 0$  with  $p > 1$  and  $b$  satisfying the growth condition  $|b(x, z, \eta)| \leq \gamma_1 |\eta|^{p-1}$ . Thus, our Theorem 1.3 generalizes the result obtained in [1, 3]. The significance of our main result in Theorem 1.2 is that it holds true for the general equation (1.1) with  $\mathbf{a}$ ,  $b$  depending on  $x$ ,  $z$  and having general structure (H1)–(H5).

Our main motivation for deriving the local gradient estimates in Theorems 1.2 and 1.3 is to be able to establish  $W^{1,q}$ -estimates (for  $q > p$ ) for weak solutions to a large class of equations of the form  $\operatorname{div} \mathbf{A}(x, u, \nabla u) + B(x, u, \nabla u) = \operatorname{div} \mathbf{F}$ , where the vector field  $\mathbf{A}$  is allowed to be discontinuous in  $x$ , Lipschitz continuous in  $u$  and its growth in the gradient variable is like the  $p$ -Laplace operator with  $1 < p < \infty$ . This is achieved in our forthcoming work [6] by using Caffarelli-Peral perturbation technique [2], and the quantified estimate (1.4) for (1.1) plays an essential role in performing that process.

The proofs of Theorems 1.2 and 1.3 will be given in section 4, after some preparations in sections 2 and 3. We prove them by employing standard iteration and interpolation techniques together with refining some results presented in [3, 7]. However, some lower order terms arising from the  $x$ ,  $z$  dependence are treated carefully and differently (see (2.2) below) compared to the known work in order to obtain the desired homogeneous estimate.

## 2. PRELIMINARY ESTIMATES

In this section we always assume that  $u$  is a weak solution of (1.1). We begin with a result which is a simple modification of [3, pages 834–835]. Throughout the paper, we denote  $w = |\nabla u|^2$  and  $|\nabla^2 u| = (\sum_{i,j=1}^n |u_{x_i x_j}|^2)^{1/2}$ .

**Lemma 2.1.** *Assume that (H2)–(H5) hold. There exists a constant  $C > 0$  depending only on  $n$ ,  $\gamma_0$  and  $\gamma_1$  such that*

$$(2.1) \quad \int_{B_3} w^{\frac{p-2}{2}} |\nabla^2 u|^2 \beta(w) \xi^2 dx + \int_{B_3} w^{\frac{p-2}{2}} |\nabla w|^2 \beta'(w) \xi^2 dx \\ \leq C \left\{ \int_{B_3} (w^{\frac{p-2}{2}} |\nabla w| + w^{\frac{p}{2}} + w^{\frac{p+1}{2}}) |\nabla \xi| \beta(w) \xi dx + \int_{B_3} (w^{\frac{p}{2}} + w^{\frac{p+2}{2}}) [\beta(w) + w \beta'(w)] \xi^2 dx \right\}$$

for any nonnegative function  $\xi \in C_0^\infty(B_3)$  and any  $\beta \in Lip_{loc}([0, \infty))$  satisfying  $\beta, \beta' \geq 0$ .

*Proof.* Using the difference-quotient argument as indicated in [13] (or [12, Proposition 1]) or using the approximation procedure as in [3], we may assume that  $u \in C^2(B_3)$  and  $|\nabla u(x)| > 0$  for every  $x \in B_3$ . For each  $i = 1, 2, \dots, n$ , define

$$(2.2) \quad \mathbf{b}_i(x, z, \eta) = \frac{\partial \mathbf{a}}{\partial x_i}(x, z, \eta) + \frac{\partial \mathbf{a}}{\partial z}(x, z, \eta) \eta_i, \quad \forall (x, z, \eta) \in B_3 \times \mathbb{K} \times \mathbb{R}^n.$$

By differentiating equation (1.1) with respect to  $x_i$ , we have

$$\operatorname{div} \left[ \sum_{j=1}^n u_{x_i x_j} \frac{\partial \mathbf{a}}{\partial \eta_j}(x, u, \nabla u) + \mathbf{b}_i(x, u, \nabla u) \right] + \frac{d}{dx_i} b(x, u, \nabla u) = 0 \quad \text{in } B_3$$

in the weak sense. Using  $\varphi = u_{x_i}\beta(w)\xi^2$  as a test function in the weak formulation and summing over  $i = 1, 2, \dots, n$ , we obtain

$$(2.3) \quad \sum_{i,j,k=1}^n \int_{B_3} \frac{\partial \mathbf{a}^k}{\partial \eta_j} u_{x_i x_j} \left[ u_{x_i x_k} \beta(w) \xi^2 + u_{x_i} w_{x_k} \beta'(w) \xi^2 + 2u_{x_i} \beta(w) \xi \xi_{x_k} \right] dx \\ = - \sum_{i=1}^n \int_{B_3} \left[ \mathbf{b}_i(x, u, \nabla u) \cdot \nabla \varphi + b(x, u, \nabla u) \varphi_{x_i} \right] dx.$$

Dealing with the LHS of (2.3), we have from assumptions (H2) and (H3) that

$$\sum_{i,j,k=1}^n \int_{B_3} \frac{\partial \mathbf{a}^k(x, u, \nabla u)}{\partial \eta_j} u_{x_i x_j} u_{x_i x_k} \beta(w) \xi^2 dx \geq \gamma_0 \sum_{i=1}^n \int_{B_3} w^{\frac{p-2}{2}} |\nabla u_{x_i}|^2 \beta(w) \xi^2 dx, \\ \sum_{i,j,k=1}^n \int_{B_3} \frac{\partial \mathbf{a}^k(x, u, \nabla u)}{\partial \eta_j} u_{x_i x_j} u_{x_i} w_{x_k} \beta'(w) \xi^2 dx \geq \frac{\gamma_0}{2} \int_{B_3} w^{\frac{p-2}{2}} |\nabla w|^2 \beta'(w) \xi^2 dx,$$

and

$$\left| \sum_{i,j,k=1}^n \int_{B_3} \frac{\partial \mathbf{a}^k(x, u, \nabla u)}{\partial \eta_j} u_{x_i x_j} u_{x_i} \beta(w) \xi \xi_{x_k} dx \right| \leq \frac{n\gamma_1}{2} \int_{B_3} w^{\frac{p-2}{2}} |\nabla w| |\nabla \xi| \beta(w) \xi dx.$$

Therefore,

$$(2.4) \quad \text{LHS of (2.3)} \geq \gamma_0 \int_{B_3} w^{\frac{p-2}{2}} |\nabla^2 u|^2 \beta(w) \xi^2 dx + \frac{\gamma_0}{2} \int_{B_3} w^{\frac{p-2}{2}} |\nabla w|^2 \beta'(w) \xi^2 dx \\ - n\gamma_1 \int_{B_3} w^{\frac{p-2}{2}} |\nabla w| |\nabla \xi| \beta(w) \xi dx.$$

For the RHS of (2.3), note that

$$\sum_{i=1}^n |\nabla \varphi_i| \leq C_n \left( |\nabla^2 u| \beta(w) \xi^2 + |\nabla u| \beta'(w) |\nabla w| \xi^2 + |\nabla u| |\nabla \xi| \beta(w) \xi \right),$$

and from (H4)–(H5) that

$$(2.5) \quad |\mathbf{b}_i(x, u, \nabla u)| + |b(x, u, \nabla u)| \leq 2\gamma_1 (w^{\frac{p-1}{2}} + w^{\frac{p}{2}}).$$

Therefore, there exists a constant  $C = C(n, \gamma_1) > 0$  such that

$$\text{RHS of (2.3)} \leq C \int_{B_3} (w^{\frac{p-1}{2}} + w^{\frac{p}{2}}) \left( |\nabla^2 u| \beta(w) \xi^2 + w^{\frac{1}{2}} \beta'(w) |\nabla w| \xi^2 + w^{\frac{1}{2}} |\nabla \xi| \beta(w) \xi \right) dx.$$

We then estimate for  $\epsilon > 0$  that

$$C \int_{B_3} (w^{\frac{p-1}{2}} + w^{\frac{p}{2}}) |\nabla^2 u| \beta(w) \xi^2 dx \leq \epsilon \int_{B_3} w^{\frac{p-2}{2}} |\nabla^2 u|^2 \beta(w) \xi^2 dx \\ + C_\epsilon \int_{B_3} (w^{\frac{p}{2}} + w^{\frac{p+2}{2}}) \beta(w) \xi^2 dx,$$

$$\begin{aligned}
C \int_{B_3} (w^{\frac{p}{2}} + w^{\frac{p+1}{2}}) \beta'(w) |\nabla w| \xi^2 dx &\leq \epsilon \int_{B_3} w^{\frac{p-2}{2}} |\nabla w|^2 \beta'(w) \xi^2 dx \\
&+ C_\epsilon \int_{B_3} (w^{\frac{p+2}{2}} + w^{\frac{p+4}{2}}) \beta'(w) \xi^2 dx.
\end{aligned}$$

Consequently,

$$\begin{aligned}
(2.6) \quad \text{RHS of (2.3)} &\leq \epsilon \int_{B_3} w^{\frac{p-2}{2}} |\nabla^2 u|^2 \beta(w) \xi^2 dx + \epsilon \int_{B_3} w^{\frac{p-2}{2}} |\nabla w|^2 \beta'(w) \xi^2 dx \\
&+ C_\epsilon \left[ \int_{B_3} (w^{\frac{p}{2}} + w^{\frac{p+2}{2}}) [\beta(w) + w \beta'(w)] \xi^2 dx + \int_{B_3} (w^{\frac{p}{2}} + w^{\frac{p+1}{2}}) |\nabla \xi| \beta(w) \xi dx \right].
\end{aligned}$$

The lemma then follows from (2.4) and (2.6) by taking  $\epsilon = \gamma_0/4$ .  $\square$

As a consequence of Lemma 2.1, we obtain:

**Lemma 2.2.** *Assume that (H2)–(H5) hold. Let  $v = w^{p/2} = |\nabla u|^p$ . Then there exists  $C = C(n, p, \gamma_0, \gamma_1) > 0$  such that*

$$\int_{B_3} |\nabla(v - k)^+|^2 \xi^2 dx \leq C \int_{B_3} [(v - k)^+]^2 |\nabla \xi|^2 dx + C \int_{B_3} (w^p + w^{p+1}) \chi_{v>k}(x) \xi^2 dx$$

for every constant  $k > 0$  and every nonnegative function  $\xi \in C_0^\infty(B_3)$ .

*Proof.* We apply Lemma 2.1 with  $\beta(s) = (s^{p/2} - k)^+$ . Then by dropping the first term in (2.1) and using  $\beta(w) + w \beta'(w) \leq (1 + \frac{p}{2}) w^{p/2} \chi_{v>k}$ , we obtain

$$\begin{aligned}
&\frac{p}{2} \int_{B_3} w^{p-2} |\nabla w|^2 \chi_{v>k}(x) \xi^2 dx \\
&\leq C \left\{ \int_{B_3} (w^{\frac{p-2}{2}} |\nabla w| + w^{\frac{p}{2}} + w^{\frac{p+1}{2}}) \xi (v - k)^+ |\nabla \xi| dx + \frac{p+2}{2} \int_{B_3} (w^p + w^{p+1}) \chi_{v>k}(x) \xi^2 dx \right\}.
\end{aligned}$$

The lemma then follows from Cauchy-Schwarz's inequality and the fact that

$$\int_{B_3} w^{p-2} |\nabla w|^2 \chi_{v>k}(x) \xi^2 dx = \frac{4}{p^2} \int_{B_3} |\nabla(v - k)^+|^2 \xi^2 dx.$$

$\square$

**Remark 2.3.** *If we assume (1.5) in place of (H4)–(H5), then (2.5) becomes*

$$|b_i(x, u, \nabla u)| + |b(x, u, \nabla u)| \leq \gamma_1 w^{\frac{p-1}{2}}.$$

Then by inspecting the proof we see that (2.1) holds without the terms  $w^{\frac{p+1}{2}}$  and  $w^{\frac{p+2}{2}}$ . As a consequence, instead of Lemma 2.2 we now obtain

$$(2.7) \quad \int_{B_3} |\nabla(v - k)^+|^2 \xi^2 dx \leq C \int_{B_3} [(v - k)^+]^2 |\nabla \xi|^2 dx + C \int_{B_3} w^p \chi_{v>k}(x) \xi^2 dx$$

for every constant  $k > 0$  and every nonnegative function  $\xi \in C_0^\infty(B_3)$ .

The next lemma gives an estimate for  $\|\nabla u\|_{L^p}$  in terms of  $\|u\|_{L^\infty}$ .

**Lemma 2.4.** *Assume that (H1)–(H3) and (H5) hold. There exists a constant  $C > 0$  depending only on  $p$ ,  $n$ ,  $\gamma_0$  and  $\gamma_1$  such that*

$$(2.8) \quad \int_{B_{\frac{r}{2}}(x_0)} |\nabla u|^p dx \leq Cr^n (r^{-p} + 1) e^{C\|u\|_{L^\infty(B_r(x_0))}} \quad \text{for every } B_r(x_0) \subset B_3.$$

*Proof.* We follow the arguments in the proof of [7, Lemma 1.1, p. 247]. Let  $M = \|u\|_{L^\infty(B_r(x_0))}$ . Since (2.8) is trivial if  $M = \infty$ , we can assume that  $M < \infty$ . Let  $\xi \in C_0^\infty(B_r(x_0))$  be the standard cut-off function with  $\xi = 1$  on  $B_{\frac{r}{2}}(x_0)$  and  $|\nabla \xi| \leq \frac{c}{r}$ . Then for any  $\lambda > 0$ , by taking  $e^{\lambda u} \xi^p$  as a test function we obtain

$$\int_{B_r(x_0)} e^{\lambda u} \left[ \lambda (\mathbf{a} \cdot \nabla u) \xi^p + p (\mathbf{a} \cdot \nabla \xi) \xi^{p-1} \right] dx = \int_{B_r(x_0)} b(x, u, \nabla u) e^{\lambda u} \xi^p dx.$$

Note that as a consequence of (H1)–(H3), we have

$$\mathbf{a}(x, u, \nabla u) \cdot \nabla u \geq \gamma_0 |\nabla u|^p \quad \text{and} \quad |\mathbf{a}(x, u, \nabla u) \cdot \nabla \xi| \xi^{p-1} \leq \gamma_1 |\nabla u|^{p-1} |\nabla \xi| \xi^{p-1}.$$

These together with condition (H5) give

$$\begin{aligned} (\lambda \gamma_0 - \gamma_1) \int_{B_r(x_0)} e^{\lambda u} |\nabla u|^p \xi^p dx &\leq \int_{B_r(x_0)} e^{\lambda u} \left( p \gamma_1 |\nabla u|^{p-1} \xi^{p-1} |\nabla \xi| + \gamma_1 |\nabla u|^{p-1} \xi^p \right) dx \\ &\leq C \int_{B_r(x_0)} e^{\lambda u} \left( |\nabla u|^p \xi^p + |\nabla \xi|^p + \xi^p \right) dx, \end{aligned}$$

where  $C$  depends only on  $p$  and  $\gamma_1$ . Choosing  $\lambda = (\gamma_1 + 2C)/\gamma_0$ , we then get

$$\int_{B_r(x_0)} |\nabla u|^p \xi^p dx \leq e^{\frac{2(\gamma_1 + 2C)M}{\gamma_0}} \int_{B_r(x_0)} (|\nabla \xi|^p + \xi^p) dx \leq e^{\frac{2(\gamma_1 + 2C)M}{\gamma_0}} (c^p r^{-p} + 1) |B_r(x_0)|.$$

This yields (2.8) as desired since  $\xi = 1$  on  $B_{\frac{r}{2}}(x_0)$ .  $\square$

We close the section by recalling a result about Hölder estimates for solutions to (1.1).

**Theorem 2.5.** ([7, Theorem 1.1, page 251]) *Assume that (H1)–(H3) and (H5) hold. Let  $u$  be a weak solution of (1.1) that satisfies (1.3). Then there exist constants  $C_0 > 0$  and  $\alpha \in (0, 1)$  depending only on  $n$ ,  $p$ ,  $\gamma_0$ ,  $\gamma_1$  and  $M_0$  such that*

$$|u(x) - u(y)| \leq C_0 |x - y|^\alpha \quad \text{for every } x, y \in B_{\frac{21}{8}}.$$

### 3. INTERPOLATION INEQUALITIES

In this section we collect some known interpolation results which will be used later. We note that they are independent of the PDE under consideration.

**Lemma 3.1.** ([7, Lemma 4.5, Chapter 2] and [3, Lemma 2.4]) *Let  $p > 1$ ,  $\rho > 0$ , and  $f \in C^2(\overline{B_\rho(x_0)})$  satisfy  $|\nabla f| > 0$ . Then for any  $\xi \in C_0^1(B_\rho(x_0))$ , we have*

$$\int_{B_\rho(x_0)} |\nabla f|^{p+2} \xi^2 dx \leq 2(\sqrt{n} + p)^2 (\text{osc}_{B_\rho(x_0)} f)^2 \int_{B_\rho(x_0)} \left[ |\nabla f|^{p-2} |\nabla^2 f|^2 \xi^2 + |\nabla f|^p |\nabla \xi|^2 \right] dx,$$

where  $\text{osc}_{B_\rho(x_0)} f = \sup_{x \in B_\rho(x_0)} |f(x) - f(x_0)|$ .

*Proof.* We include a proof for the sake of completeness. Let  $v = |\nabla f|^2$ . Then

$$\int_{B_\rho(x_0)} v^{\frac{p+2}{2}} \xi^2 dx = \int_{B_\rho(x_0)} v^{\frac{p}{2}} |\nabla f|^2 \xi^2 dx = \int_{B_\rho(x_0)} v^{\frac{p}{2}} f_{x_i} [f(x) - f(0)]_{x_i} \xi^2 dx.$$

Therefore, the integration by parts yields

$$\begin{aligned} \int_{B_\rho(x_0)} v^{\frac{p+2}{2}} \xi^2 dx &= - \int_{B_\rho(x_0)} [f(x) - f(x_0)] \left[ v^{\frac{p}{2}} \Delta f \xi^2 + p v^{\frac{p-2}{2}} f_{x_i} f_{x_l} f_{x_l x_i} \xi^2 + 2 v^{\frac{p}{2}} f_{x_i} \xi \xi_{x_i} \right] dx \\ &\leq \text{osc}_{B_\rho(x_0)} f \int_{B_\rho(x_0)} \left[ (\sqrt{n} + p) v^{\frac{p}{2}} |\nabla^2 f| \xi^2 + 2 v^{\frac{p+1}{2}} |\nabla \xi| \xi \right] dx \\ &\leq \frac{1}{2} \int_{B_\rho(x_0)} v^{\frac{p+2}{2}} \xi^2 dx + (\sqrt{n} + p)^2 (\text{osc}_{B_R} f)^2 \int_{B_R} \left[ v^{\frac{p-2}{2}} |\nabla^2 f|^2 \xi^2 + v^{\frac{p}{2}} |\nabla \xi|^2 \right] dx. \end{aligned}$$

The lemma then follows.  $\square$

The next interpolation result is extracted from [1, page 55].

**Lemma 3.2.** *Let  $f \in L^\infty(B_R)$  with  $R > 0$ . Assume that there exist constants  $q > p > 0$  and  $\gamma > 0$  such that*

$$(3.1) \quad \|f\|_{L^\infty(B_{(1-\sigma)r})} \leq \frac{\gamma}{(\sigma r)^{\frac{n}{q}}} \left( \int_{B_r} |f|^q dx \right)^{\frac{1}{q}}$$

for every  $r \in (0, R)$  and every  $\sigma \in (0, 1)$ . Then we have

$$\|f\|_{L^\infty(B_{\frac{R}{2}})} \leq \frac{\gamma'}{R^{\frac{n}{p}}} \left( \int_{B_R} |f|^p dx \right)^{\frac{1}{p}},$$

where  $\gamma' = \frac{p}{q-p} 2^{\frac{n}{p}} \left( 2^{\frac{n}{p}+1} \frac{q-p}{q} \gamma \right)^{\frac{q}{p}}$ . In particular,  $\gamma' = 8^{\frac{n}{p}} \gamma^2$  if  $q = 2p$ .

*Proof.* The proof of this lemma for particular  $q = p + 2$  is in [1, page 55]. For the sake of completeness, we include the same arguments for all  $q > p$  here.

Let  $G = \left( \int_{B_R} |f|^p dx \right)^{\frac{1}{p}}$ , and for  $s = 0, 1, \dots$ ,

$$r_s = \frac{R}{2} \sum_{i=0}^s 2^{-i}, \quad F_s = \|f\|_{L^\infty(B_{r_s})}.$$

Then by applying (3.1) to  $r = r_{s+1}$  and  $\sigma r = r_{s+1} - r_s = R/2^{s+2}$ , we obtain that

$$F_s \leq 2^{\frac{ns}{q}} \left( \frac{4}{R} \right)^{\frac{n}{q}} \gamma \left( \int_{B_{r_{s+1}}} |f|^q dx \right)^{\frac{1}{q}} \leq 2^{\frac{ns}{q}} \left( \frac{4}{R} \right)^{\frac{n}{q}} \gamma F_{s+1}^{\frac{q-p}{q}} G^{\frac{p}{q}}.$$

Using Young's inequality, it follows for any  $\delta > 0$  that

$$(3.2) \quad F_s \leq \delta F_{s+1} + 2^{\frac{ns}{p}} \Theta G \quad \text{for } s = 0, 1, \dots$$

with  $\Theta = \frac{p}{q} \left( \frac{q-p}{\delta q} \right)^{\frac{q-p}{q}} \left( \frac{4}{R} \right)^{\frac{n}{p}} \gamma^{\frac{q}{p}}$ . Thus by iterating the relation (3.2), we get

$$F_0 \leq \delta^s F_s + \Theta G \sum_{i=0}^{s-1} (\delta 2^{\frac{n}{p}})^i \leq \delta^s \|f\|_{L^\infty(B_R)} + \Theta G \sum_{i=0}^{s-1} (\delta 2^{\frac{n}{p}})^i$$

for any  $s = 1, 2, \dots$ . Then by choosing  $\delta = 2^{-(\frac{n}{p}+1)}$  and letting  $s \rightarrow \infty$ , we deduce that

$$F_0 \leq 2\Theta G = \frac{2p}{q} \left( 2^{\frac{n}{p}+1} \frac{q-p}{q} \right)^{\frac{q-p}{q}} \left( \frac{4}{R} \right)^{\frac{n}{p}} \gamma^{\frac{q}{p}} G = \frac{\gamma'}{R^{\frac{n}{p}}} G.$$

This completes the proof as  $F_0 = \|f\|_{L^\infty(B_{\frac{R}{2}})}$ .  $\square$

#### 4. PROOFS OF MAIN THEOREMS

We start with proving Theorem 1.2. Our proof consists of two main steps, and the crucial one is given in the following proposition.

**Proposition 4.1.** *Assume that (H2)–(H5) hold. Let  $u$  be a weak solution of (1.1) that satisfies*

$$(4.1) \quad \int_{B_{\frac{5}{2}}} |\nabla u|^{2(p+\bar{q})} dx \leq \bar{M} \quad \text{for some } \bar{q} > \max\{1, \frac{n}{2}\}.$$

*Then there exists  $C > 0$  depending only on  $n, p, \bar{q}, \gamma_0, \gamma_1$ , and  $\bar{M}$  such that inequality (1.4) holds true.*

*Proof.* The proof uses Lemma 2.2 and De Giorgi's iteration. We provide full calculations here. Without loss of generality, we assume  $x_0 = 0$ .

Let  $v = w^{p/2} = |\nabla u|^p$ . For each  $k > 0$  and  $r > 0$ , denote

$$A_{k,r} = \{x \in B_r : v(x) > k\}.$$

Let  $K$  be a positive number which will be determined. Let  $\zeta(s)$  be a smooth cut-off function on  $\mathbb{R}$  which equals unity for  $s \leq 0$ , vanishes for  $s \geq \frac{1}{2}$ , and  $|\zeta'| \leq c$  for some constant  $c > 0$ .

Let us fix  $R \in (0, 3/2]$  and  $\sigma \in (0, 1)$ . Then for  $i = 0, 1, 2, \dots$ , we denote

$$\rho_i = \left(1 - \sigma + \frac{\sigma}{2^i}\right)R, \quad \bar{\rho}_i = \frac{\rho_i + \rho_{i+1}}{2}, \quad \xi_i(y) = \zeta\left(\frac{2^{i+1}}{\sigma R}(|y| - \rho_{i+1})\right),$$

$$k_i = K\left(1 - \frac{1}{2^i}\right), \quad v_i = (v - k_i)^+.$$

Then  $\rho_{i+1} < \bar{\rho}_i < \rho_i$ , the function  $\xi_i$  vanishes outside  $B_{\bar{\rho}_i}$ , equals unity on  $B_{\rho_{i+1}}$ , and

$$(4.2) \quad 0 \leq \xi_i \leq 1, \quad |\nabla \xi_i| \leq \frac{c 2^{i+1}}{\sigma R} \quad \text{on } B_3.$$



Let  $n/2 < q \leq \infty$ . By applying Lemma 2.2 with  $k = k_{i+1} > 0$ ,  $\xi = \xi_i$  and by using (4.2) together with Hölder's inequality, we obtain

$$\begin{aligned}
 (4.3) \quad \int_{B_3} |\nabla(v_{i+1}\xi_i)|^2 dx &\leq C \int_{B_3} v_{i+1}^2 |\nabla \xi_i|^2 dx + C \int_{B_3} (w^p + w^{p+1}) \chi_{v > k_{i+1}}(x) \xi_i^2 dx \\
 &\leq C \left[ \frac{4^i}{(\sigma R)^2} \int_{A_{k_{i+1}, \rho_i}} v_{i+1}^2 dx + \int_{A_{k_{i+1}, \rho_i}} w^p dx + \int_{A_{k_{i+1}, \rho_i}} w^{\frac{p}{q}+1} w^{\frac{p}{q'}} dx \right] \\
 &\leq C \left\{ \left( \int_{A_{k_{i+1}, \rho_i}} |\nabla u|^{2p} dx \right)^{\frac{1}{q}} \left[ \frac{4^i}{(\sigma R)^2} \left( \int_{A_{k_{i+1}, \rho_i}} v_{i+1}^2 dx \right)^{\frac{1}{q'}} + \left( \int_{A_{k_{i+1}, \rho_i}} w^p dx \right)^{\frac{1}{q'}} \right] \right. \\
 &\quad \left. + \left( \int_{A_{k_{i+1}, \rho_i}} |\nabla u|^{2(p+q)} dx \right)^{\frac{1}{q}} \left( \int_{A_{k_{i+1}, \rho_i}} w^p dx \right)^{\frac{1}{q'}} \right\}.
 \end{aligned}$$

Let  $M_q(R) = \| |\nabla u|^{\frac{2p}{q}} \|_{L^q(B_R)} + \| |\nabla u|^{\frac{2p}{q}+2} \|_{L^q(B_R)}$  with the convention that  $M_\infty(R) = 1 + \| \nabla u \|_{L^\infty(B_R)}^2$ . Then it follows from (4.3) that

$$\begin{aligned}
 (4.4) \quad \int_{B_3} |\nabla(v_{i+1}\xi_i)|^2 dx &\leq CM_q(R) \left\{ \frac{4^i}{(\sigma R)^2} \left( \int_{A_{k_{i+1}, \rho_i}} v_{i+1}^2 dx \right)^{\frac{1}{q'}} + \left( \int_{A_{k_{i+1}, \rho_i}} v^2 dx \right)^{\frac{1}{q'}} \right\} \\
 &\leq CM_q(R) \left\{ \frac{4^i}{(\sigma R)^2} \left( \int_{A_{k_{i+1}, \rho_i}} v_{i+1}^2 dx \right)^{\frac{1}{q'}} + \left( \int_{A_{k_{i+1}, \rho_i}} v_{i+1}^2 dx + K^2 |A_{k_{i+1}, \rho_i}| \right)^{\frac{1}{q'}} \right\} \\
 &\leq CM_q(R) \left\{ \frac{4^i}{(\sigma R)^2} \left( \int_{A_{k_{i+1}, \rho_i}} v_{i+1}^2 dx \right)^{\frac{1}{q'}} + \left( K^2 |A_{k_{i+1}, \rho_i}| \right)^{\frac{1}{q'}} \right\},
 \end{aligned}$$

where  $C$  depends only on  $n, p, \gamma_0, \gamma_1$ .

We next show that (4.4) implies the desired estimate (1.4). For this, let us define

$$J_i = \int_{A_{k_i, \rho_i}} v_i^2 dx.$$

By properties of  $\xi_i$ , Sobolev's embedding  $W^{1, \frac{2n}{n+2}}(B_3) \hookrightarrow L^2(B_3)$  when  $n \geq 2$ , and Hölder's inequality, we have

$$J_{i+1} \leq \int_{B_3} (v_{i+1}\xi_i)^2 dx \leq C \left( \int_{B_3} |\nabla(v_{i+1}\xi_i)|^{\frac{2n}{n+2}} dx \right)^{\frac{n+2}{n}} \leq C |A_{k_{i+1}, \rho_i}|^{\frac{2}{n}} \int_{B_3} |\nabla(v_{i+1}\xi_i)|^2 dx.$$

We note that this estimate for  $J_{i+1}$  still holds true when  $n = 1$ . Indeed, in that case we can use the Sobolev's embedding  $W^{n,1}(B_3) \hookrightarrow C(\overline{B_3})$  and Hölder's inequality to obtain

$$\begin{aligned}
 J_{i+1} &\leq \int_{B_3} (v_{i+1}\xi_i)^2 dx \leq |A_{k_{i+1}, \rho_i}| \|v_{i+1}\xi_i\|_{L^\infty(B_3)}^2 \leq C |A_{k_{i+1}, \rho_i}| \left( \int_{B_3} |\nabla(v_{i+1}\xi_i)| dx \right)^2 \\
 &\leq C |A_{k_{i+1}, \rho_i}|^2 \int_{B_3} |\nabla(v_{i+1}\xi_i)|^2 dx.
 \end{aligned}$$

It follows from the estimate for  $J_{i+1}$ , (4.4) and the fact  $\int_{A_{k_{i+1}, \rho_i}} v_{i+1}^2 dx \leq \int_{A_{k_{i+1}, \rho_i}} v_i^2 dx \leq J_i$  that

$$(4.5) \quad J_{i+1} \leq CM_q(R) |A_{k_{i+1}, \rho_i}|^{\frac{2}{n}} \left[ \frac{4^i}{(\sigma R)^2} J_i^{\frac{1}{q'}} + (K^2 |A_{k_{i+1}, \rho_i}|)^{\frac{1}{q'}} \right].$$

The monotonicity of  $k_i$  implies that

$$J_i \geq \int_{A_{k_{i+1}, \rho_i}} (v - k_i)^2 dx \geq (k_{i+1} - k_i)^2 |A_{k_{i+1}, \rho_i}| = 4^{-(i+1)} K^2 |A_{k_{i+1}, \rho_i}|,$$

which gives

$$(4.6) \quad |A_{k_{i+1}, \rho_i}| \leq 4^{i+1} K^{-2} J_i.$$

From (4.5) and (4.6), we deduce that

$$(4.7) \quad J_{i+1} \leq CM_q(R) |A_{k_{i+1}, \rho_i}|^{\frac{2}{n}} \frac{4^i}{(\sigma R)^2} J_i^{\frac{1}{q'}} \leq CM_q(R) (\sigma R)^{-2} K^{\frac{-4}{n}} B^i J_i^{1+\kappa},$$

where  $B = 4^{\frac{2}{n}+1}$  and  $\kappa = \frac{2}{n} - \frac{1}{q}$ . Note that as  $q > n/2$ , we have  $\kappa > 0$ .

By iterating formula (4.7), we see that

$$J_i \leq [CM_q(R) (\sigma R)^{-2} K^{\frac{-4}{n}}]^{\frac{(1+\kappa)^i - 1}{\kappa}} B^{\frac{(1+\kappa)^i - 1}{\kappa}} J_0^{(1+\kappa)^i} \quad \text{for all } i = 0, 1, \dots$$

Next, select

$$K = [CM_q(R) (\sigma R)^{-2}]^{\frac{n}{4}} B^{\frac{n}{4\kappa}} \left[ \int_{B_R} |\nabla u|^{2p} dx \right]^{\frac{n\kappa}{4}}$$

which ensures

$$J_0 = \int_{A_{k_0, \rho_0}} |\nabla u|^{2p} dx \leq \int_{B_R} |\nabla u|^{2p} dx = [CM_q(R) (\sigma R)^{-2} K^{\frac{-4}{n}}]^{-\frac{1}{\kappa}} B^{-\frac{1}{\kappa^2}} =: \Lambda.$$

Therefore, we obtain

$$J_i \leq \Lambda B^{-\frac{i}{\kappa}} \rightarrow 0, \quad \text{as } i \rightarrow \infty.$$

Hence, we conclude that

$$|\nabla u(x)|^p = v(x) \leq K \quad \text{a.e. in } B_{(1-\sigma)R}.$$

Thus we have proved that

$$(4.8) \quad |\nabla u(x)| \leq \frac{1}{(\sigma R)^{\frac{n}{2p}}} \left[ CB^{\frac{1}{2}} M_q(R) \left( \int_{B_R} |\nabla u|^{2p} dx \right)^{\kappa} \right]^{\frac{n}{4p}} \quad \text{a.e. in } B_{(1-\sigma)R}$$

for every  $R \in (0, 3/2]$ ,  $\sigma \in (0, 1)$  and  $2/n < q \leq \infty$ . By taking  $R = 3/2$ ,  $\sigma = 1/3$ ,  $q = \bar{q}$  and using assumption (4.1), we see that the right hand side of (4.8) is bounded. As a consequence, there exists a constant  $C_*$  depending only on  $n, p, \bar{q}, \gamma_0, \gamma_1$  and  $\bar{M}$  such that

$$(4.9) \quad \|\nabla u\|_{L^\infty(B_1)} \leq C_*.$$

Next, we infer from (4.8) with  $q = \infty$ , the fact  $\kappa = \frac{2}{n} - \frac{1}{q}$  and (4.9) that

$$\|\nabla u\|_{L^\infty(B_{(1-\sigma)R})} \leq \frac{[CB^{\frac{n}{2}} M_\infty(1)]^{\frac{n}{4p}}}{(\sigma R)^{\frac{n}{2p}}} \left( \int_{B_R} |\nabla u|^{2p} dx \right)^{\frac{1}{2p}} \leq \frac{\gamma}{(\sigma R)^{\frac{n}{2p}}} \left( \int_{B_R} |\nabla u|^{2p} dx \right)^{\frac{1}{2p}}$$

for every  $R \in (0, 1]$  and every  $\sigma \in (0, 1)$ , where  $\gamma = [CB^{\frac{n}{2}} (1 + C_*^2)]^{\frac{n}{4p}}$ . Hence, we can use the interpolation result in Lemma 3.2 for  $q = 2p$  to get

$$\|\nabla u\|_{L^\infty(B_{\frac{R}{2}})} \leq \frac{8^{\frac{n}{p}} \gamma^2}{R^{\frac{n}{p}}} \left( \int_{B_R} |\nabla u|^p dx \right)^{\frac{1}{p}} \quad \text{for all } 0 < R \leq 1.$$

Therefore, the proof is complete.  $\square$

We are now ready to prove our main results.

**Proof of Theorem 1.2.** Thanks to Proposition 4.1, it remains to verify condition (4.1). We complete this step by claiming that for any positive integer  $m$  there exists a constant  $M > 0$  depending only on  $n, p, m, \gamma_0, \gamma_1$  and  $M_0$  such that

$$(4.10) \quad \int_{B_{\frac{5}{2}}} |\nabla u|^{p+2m} dx \leq M.$$

Indeed, let us fix  $m \in \{1, 2, \dots\}$ . As in the proof of Lemma 2.1, we can assume that  $u \in C^2(B_3)$  with  $|\nabla u| > 0$ . Let  $x_0 \in B_{\frac{5}{2}}$  and  $0 < \rho \leq 1/8$  be arbitrary, which ensure that  $B_{2\rho}(x_0) \subset B_{\frac{11}{4}}$ . Consider  $s \geq 0$  and a nonnegative function  $\xi \in C_0^\infty(B_\rho(x_0))$ . Then by applying Lemma 2.1 with  $\beta(w) = (w + \delta)^s$  and letting  $\delta \rightarrow 0^+$ , we obtain

$$\begin{aligned} I &\stackrel{\text{def}}{=} \int_{B_3} w^{\frac{p-2+2s}{2}} |\nabla^2 u|^2 \xi^2 dx + s \int_{B_3} w^{\frac{p-4+2s}{2}} |\nabla w|^2 \xi^2 dx \\ &\leq C(s+1) \int_{B_3} (w^{\frac{p+2s}{2}} + w^{\frac{p+2+2s}{2}}) \xi^2 dx \\ &\quad + C \int_{B_3} w^{\frac{p-1+2s}{2}} |\nabla^2 u| |\nabla \xi| \xi dx + \int_{B_3} (w^{\frac{p+2s}{2}} + w^{\frac{p+1+2s}{2}}) |\nabla \xi| \xi dx \end{aligned}$$

where we have used  $|\nabla w| \leq Cw^{\frac{1}{2}} |\nabla^2 u|$ . It follows from Young's inequality and by moving some terms around that

$$(4.11) \quad I \leq C(s+1) \int_{B_3} (w^{\frac{p+2s}{2}} + w^{\frac{p+2+2s}{2}}) \xi^2 dx + C \int_{B_3} w^{\frac{p+2s}{2}} |\nabla \xi|^2 dx$$

with  $C$  depending only on  $n, \gamma_0$  and  $\gamma_1$ . Next, applying Lemma 3.1 for  $f = u$  and with test function  $w^{s/2} \xi$ , we get

$$(4.12) \quad \int_{B_3} w^{\frac{p+2+2s}{2}} \xi^2 dx \leq 4(\sqrt{n} + p)^2 (\text{osc}_{B_\rho(x_0)} u)^2 \left[ (s+1)I + \int_{B_3} w^{\frac{p+2s}{2}} |\nabla \xi|^2 dx \right].$$

Owing to assumption (1.3) and the fact  $B_\rho(x_0) \subset B_{\frac{21}{8}}$ , we can infer from Theorem 2.5 that  $\text{osc}_{B_\rho(x_0)} u \leq C_0 \rho^\alpha$ . Thus we deduce from (4.12) and (4.11) that

$$(4.13) \quad \int_{B_3} w^{\frac{p+2+2s}{2}} \xi^2 dx \leq \gamma \rho^{2\alpha} (s+1)^2 \left[ \int_{B_3} w^{\frac{p+2+2s}{2}} \xi^2 dx + \int_{B_3} w^{\frac{p+2s}{2}} (\xi^2 + |\nabla \xi|^2) dx \right],$$

where  $\gamma$  and  $\alpha$  depend only on  $n, p, \gamma_0, \gamma_1$  and  $M_0$ . Now let  $R_0 = \min\{(2\gamma)^{-\frac{1}{2\alpha}}, \frac{1}{8}\}$ , and

$$R_s = R_0(1+s)^{-\frac{1}{\alpha}} \quad \text{for } s \geq 0.$$

Let  $\xi_s \in C_0^\infty(B_{R_s}(x_0))$  be the standard cut-off function which equals one in  $B_{R_{s+1}}(x_0)$ , and

$$|\nabla \xi_s| \leq \frac{2}{R_s - R_{s+1}}.$$

Then by using this test function in (4.13), we obtain

$$\int_{B_{R_s}(x_0)} w^{\frac{p+2+2s}{2}} \xi_s^2 dx \leq \frac{1}{2} \int_{B_{R_s}(x_0)} w^{\frac{p+2+2s}{2}} \xi_s^2 dx + \frac{1}{2} \int_{B_{R_s}(x_0)} w^{\frac{p+2s}{2}} (\xi_s^2 + |\nabla \xi_s|^2) dx$$

yielding

$$\int_{B_{R_{s+1}}(x_0)} |\nabla u|^{p+2s+2} dx \leq \frac{5}{(R_s - R_{s+1})^2} \int_{B_{R_s}(x_0)} |\nabla u|^{p+2s} dx \quad \forall s \geq 0.$$

By iterating this inequality from  $s = 0$  to  $s = m - 1$  and using the fact  $R_{i-1} - R_i \geq (1+i)^{-\frac{2}{\alpha}}$ , we conclude that

$$\int_{B_{R_m}(x_0)} |\nabla u|^{p+2m} dx \leq \frac{5^m}{\prod_{i=1}^m (R_{i-1} - R_i)^2} \int_{B_{R_0}(x_0)} |\nabla u|^p dx \leq 5^m [(m+1)!]^{\frac{2}{\alpha}} \int_{B_{R_0}(x_0)} |\nabla u|^p dx.$$

As  $B_{2R_0}(x_0) \subset B_{\frac{11}{4}}$ , we can use Lemma 2.4 together with assumption (1.3) to bound the above right-hand side. Consequently, we obtain

$$(4.14) \quad \int_{B_{R_m}(x_0)} |\nabla u|^{p+2m} dx \leq C(n, p, m, \gamma_0, \gamma_1, M_0) \quad \text{for all } x_0 \in B_{\frac{5}{2}}.$$

Now by covering  $B_{\frac{5}{2}}$  with a finite number of balls  $B_{R_m}(x_i)$  with  $x_i \in B_{\frac{5}{2}}$ , we deduce claim (4.10) from (4.14). The proof is therefore complete.  $\square$

**Proof of Theorem 1.3.** The proof is a direct consequence of that of Proposition 4.1. Observe that in the proof of Proposition 4.1, assumption (4.1) is only used to control the term  $w^{p+1}$  in (4.3) which comes from Lemma 2.2. Thus by using (2.7) in place of Lemma 2.2, we see that (4.4) holds for  $q = \infty$  and with  $M_\infty(R)$  being replaced by 1. Therefore, estimate (4.7) is valid without the term  $M_q(R)$  and for  $\kappa = 2/n$ . With this change and by repeating the arguments after (4.7), we obtain (1.4). Note also that assumption (H1) is not needed since Lemma 2.4, Theorem 2.5 and Lemma 3.1 are not used in the proof.  $\square$

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